UK Maths Trust

Intermediate Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

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Solutions and investigations

31 January 2024

These solutions augment the shorter solutions also available online. The shorter solutions in many cases omit details. The solutions given here are full solutions, as explained below. In some cases alternative solutions are given. There are also many additional problems for further investigation. We welcome comments on these solutions and the additional problems. Please send them to challenges@ukmt.org.uk.

The Intermediate Mathematical Challenge (IMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that sometimes you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the IMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So for each question we have included a complete solution which does not use the fact that one of the given alternatives is correct. Thus we have aimed to give full solutions with each step explained (or, occasionally, left as an exercise). We therefore hope that these solutions can be used as a model for the type of written solution that is expected when a complete solution to a mathematical problem is required (for example, in the Intermediate Mathematical Olympiad and similar competitions).

Enquiries about the Intermediate Mathematical Challenge should be sent to:

IMC, challenges@ukmt.org.uk

www.ukmt.org.uk

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
D	A	A	D	C	C	B	C	D	B	C	D	C	B	B	D	E	E	A	D	E	E	B	C	E

1. What is the value of $\dfrac{20+24}{20-24}$?

A 8 B −9 C 10 D −11 E 12

SOLUTION **D**

This is not a difficult calculation, but it becomes even easier if you notice that a common factor of 4 can be cancelled in the numerator and the denominator.

We have

$$\frac{20+24}{20-24} = \frac{5+6}{5-6} = \frac{11}{-1} = -11.$$

FOR INVESTIGATION

1.1 What is the value of $\dfrac{20\,000+24\,000}{20\,000-24\,000}$?

1.2 What is the value of $\dfrac{20\times 24}{20\div 24}$?

1.3 Solve the equation $\dfrac{20+x}{20-x} = 24$.

2. What is the difference between the smallest two-digit prime and the largest two-digit prime?

A 86 B 84 C 82 D 80 E 78

SOLUTION **A**

99 and 98 are not primes, but 97 is a prime. Therefore the largest two-digit prime is 97.

10 is not prime but 11 is a prime. Therefore the smallest two-digit prime is 11.

The difference between these is 97 − 11 which is equal to 86.

FOR INVESTIGATION

2.1 Show that 99 and 98 are not primes, and that 97 is prime.

2.2 List all the two-digit primes.

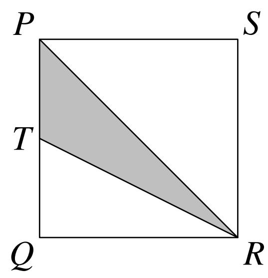
2.3 What is the difference between the second smallest two-digit prime and the second largest two-digit prime?

2.4 What is the difference between the smallest three-digit prime and the largest three-digit prime?

2.5 Find the smallest prime p for which there exists a prime q with $q - p = 86$.

3. The diagram shows the square $PQRS$ and T, the midpoint of the side PQ.

What fraction of the area of the square $PQRS$ is shaded?



A $\frac{1}{4}$ B $\frac{1}{3}$ C $\frac{1}{2}$ D $\frac{2}{3}$ E $\frac{3}{4}$

Solution **A**

We use the formula

$$\text{area} = \tfrac{1}{2}(\text{base} \times \text{height}),$$

for the area of a triangle.

With base PT the height of the triangle PTR equals the length of QR. Therefore

$$\text{area of the triangle } PTR = \tfrac{1}{2}(PT \times QR).$$

Since T is the midpoint of PQ, we have $PT = \frac{1}{2}PQ$. Therefore

$$\begin{aligned}\text{area of triangle } PTR &= \tfrac{1}{2}(\tfrac{1}{2}PQ \times QR)\\ &= \tfrac{1}{4}(PQ \times QR)\\ &= \tfrac{1}{4}(\text{area of } PQRS).\end{aligned}$$

Therefore the fraction of the square $PQRS$ that is shaded is $\frac{1}{4}$.

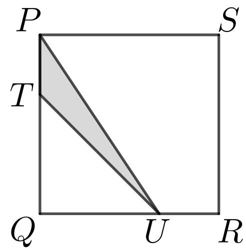
For investigation

3.1 $PQRS$ is a square.

The point T divides PQ in the ratio 1 : 2.

The point U divides QR in the ratio 2 : 1.

What is the area of the triangle PUT as a fraction of the area of the square $PQRS$?



3.2 Explain why the formula

$$\text{area} = \tfrac{1}{2}(\text{base} \times \text{height}),$$

for the area of a triangle is correct.

4. The shorter sides of a right-angled triangle have lengths $\sqrt{5}$ and $\sqrt{12}$.

What is the length of the hypotenuse?

A $\sqrt{7}$ B $\sqrt{13}$ C $\sqrt{15}$ D $\sqrt{17}$ E 7

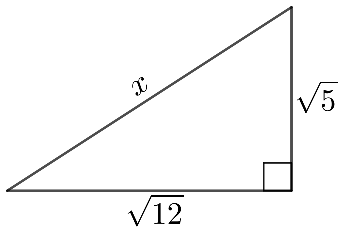
Solution **D**

Let x be the length of the hypotenuse.

By Pythagoras' Theorem,

$$x^2 = \sqrt{12}^2 + \sqrt{5}^2 = 12 + 5 = 17.$$

Therefore $x = \sqrt{17}$.

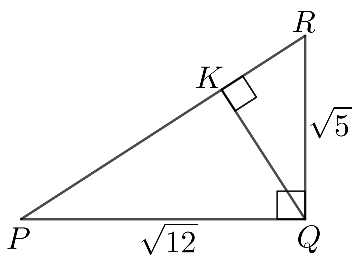


For investigation

4.1 In the right-angled triangle PQR, $PQ = \sqrt{12}$ and $RQ = \sqrt{5}$.

K is the point on PR such that QK is perpendicular to PR.

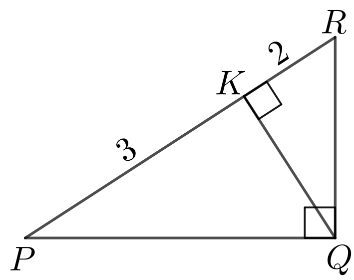
What is the length of KR?



4.2 In the right-angled triangle PQR, K is the point on PR such that QK is perpendicular to PR.

$PK = 3$ and $KR = 2$.

What are the lengths of PQ and QR?



4.3 If you have not seen a proof of Pythagoras' Theorem before, find one by looking in a book, or by searching the internet, or by asking your teacher.

5. The ages of Grannie's seven grandchildren are consecutive positive integers. The youngest three grandchildren have a mean age of 6.

What is the mean age of the oldest three grandchildren?

A 8 B 9 C 10 D 11 E 12

Solution **C**

Method 1

Because the mean age of the three youngest grandchildren is 6, and their ages are consecutive integers, their ages are 5, 6 and 7.

It follows that the ages of all the seven grandchildren are 5, 6, 7, 8, 9, 10, 11, So the ages of the three oldest grandchildren are 9, 10, 11.

Hence their mean age is 10.

Method 2

We number Grannie's seven grandchildren as 1, 2, 3, 4, 5, 6 and 7, from youngest to oldest. Because their ages are consecutive integers, child 5 is 4 years older than child 1, child 6 is 4 years older than child 2, and child 7 is 4 years older than child 3.

Therefore the mean age of children 5, 6 and 7 is 4 years more than the mean age of children 1, 2 and 3.

Hence the mean age of the three oldest children is $6 + 4 = 10$.

For investigation

5.1 In this question, let n be the age of the youngest grandchild.

Use the fact that the mean age of the three youngest grandchildren is 6 to find an equation involving n. Hence find the value of n and use this to obtain the mean age of the three oldest grandchildren.

5.2 The mean of the three largest numbers in a sequence of nine consecutive integers is 27.

What is the mean of all the numbers in the sequence?

5.3 The mean of the three smallest numbers in a sequence of consecutive integers is 6. The mean of the three largest numbers in this sequence is 15.

How many integers are there in this sequence?

6. Four of these points lie on a circle. Which of the points does not lie on that circle?

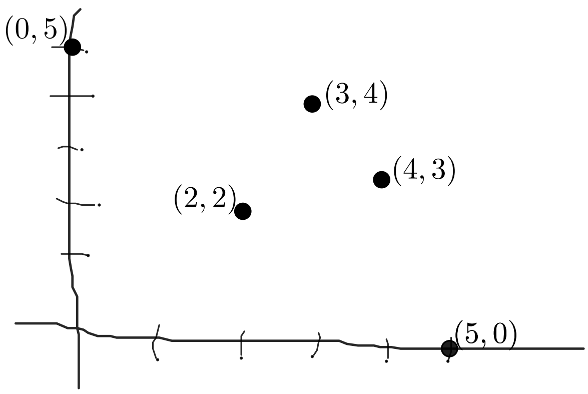
A (5, 0) B (4, 3) C (2, 2) D (3, 4) E (0, 5)

Solution **C**

Method 1

A rough sketch suggests that the points $(5,0)$, $(4,3)$, $(3,4)$ and $(0,5)$ lie on the circle with centre the origin $(0,0)$, and radius 5, whereas the point $(2,2)$ does not lie on this circle.

Even though it is a very rough sketch, in the context of the IMC it is now safe to pick out C as the correct option.

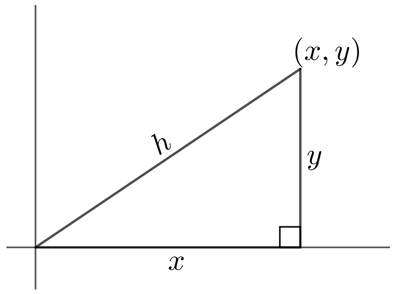


Method 2

In this method we give the calculations that are needed to prove that our answer above is correct.

From Pythagoras' Theorem, we see that the distance of the point (x, y) from the origin $(0,0)$ is h where $h^2 = x^2 + y^2$, and therefore

$$h = \sqrt{x^2 + y^2}.$$



Using this formula, we obtain the following table.

point	distance from (0,0)
A $(5,0)$	$\sqrt{5^2+0^2} = \sqrt{25+0} = \sqrt{25} = 5$
B $(4,3)$	$\sqrt{4^2+3^2} = \sqrt{16+9} = \sqrt{25} = 5$
C $(2,2)$	$\sqrt{2^2+2^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$
D $(3,4)$	$\sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$
E $(0,5)$	$\sqrt{0^2+5^2} = \sqrt{0+25} = \sqrt{25} = 5$

Hence C is the only one of the given points not on the circle with centre $(0,0)$ and radius 5.

For investigation

6.1 Give examples of more points that lie on the same circle.

7. The ‘Penny’s Puddings’ company uses one tonne of rice to make twenty-five thousand cans of rice pudding. Each tonne of rice contains approximately fifty million grains of rice.

Approximately how many grains of rice are there in a can of Penny’s rice pudding?

A 200 B 2000 C 5000 D 50 000 E 1 250 000

SOLUTION **B**

The approximate number of grains of rice in each can is

$$\frac{\text{fifty million}}{\text{twenty-five thousand}} = \frac{50\,000\,000}{25\,000} = \frac{50\,000}{25} = 2000.$$

FOR INVESTIGATION

7.1 The weight of a rice pudding is about ten times the weight of the dry rice that is used in making it.

What is the approximate weight, in grams, of the rice pudding in a single can of Penny’s rice pudding?

8. What is the value of $999 \times 999 + 999$?

A 10 800 B 100 800 C 999 000 D 999 999 E 1 000 998

SOLUTION **C**

Without the use of a calculator, you need to avoid calculating the value of 999×999 if at all possible. You can do this by taking out 999 as a common factor of both 999×999 and 999 $(= 999 \times 1)$. Then, instead of multiplying by 999, the answer can be found by multiplying by 1000 which is easy to do without a calculator.

$$999 \times 999 + 999 = 999 \times (999 + 1) = 999 \times 1000 = 999\,000.$$

FOR INVESTIGATION

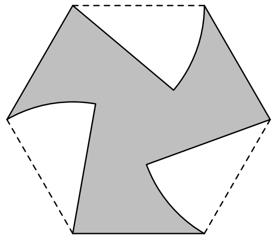
8.1 What is the value of $9999 \times 9999 + 9999$?

8.2 What is the value of $13\,579 \times 2468 + 13\,579 \times 7532$?

8.3 Find the solution of the equation

$$2468 \times 97\,531 + 2468x = 246\,800\,000.$$

9. Three sectors of a circle are removed from a regular hexagon to form the shaded shape shown. Each sector has perimeter 18 mm.

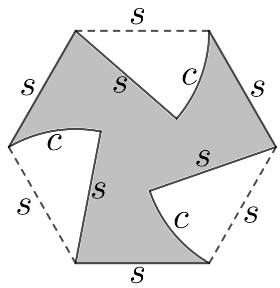


What is the perimeter, in mm, of the shaded shape formed?

A 48 B 50 C 52 D 54 E 56

SOLUTION **D**

We let s be the length, in mm, of the sides of the hexagon.



Each of the three sectors has two straight edges of length s. Because each of these sectors has the same perimeter, it follows that the arcs that are parts of the perimeters of these sectors all have the same length. We let c be this length, in mm.

The perimeter of each sector is 18mm. Therefore $2s + c = 18$.

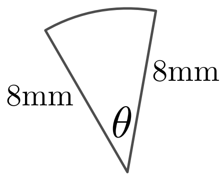
We see from the diagram that the perimeter of the shaded shape is made up of six straight edges and three arcs.

It follows that the perimeter of the shaded shape, in mm, is given by

$$6s + 3c = 3(2s + c) = 3 \times 18 = 54.$$

FOR INVESTIGATION

9.1 Suppose that the side length of the regular hexagon of this question is 8mm, and, as in the question, the perimeter of each of the sectors is 18mm.



(a) In this case, what is the angle θ in each sector?

(b) In this case, what is the area of the shaded shape?

10. Which of these is equal to $\frac{20}{24} + \frac{20}{25}$?

A $\frac{40}{600}$ B $\frac{49}{30}$ C $\frac{30}{49}$ D $\frac{40}{49}$ E $\frac{49}{40}$

SOLUTION **B**

We have

$$\frac{20}{24} + \frac{20}{25} = \frac{5}{6} + \frac{4}{5} = \frac{25}{30} + \frac{24}{30} = \frac{49}{30}.$$

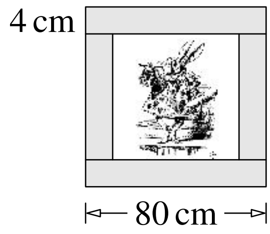
FOR INVESTIGATION

10.1 What is the value of $\frac{20}{25} + \frac{20}{26}$?

11. A picture, together with its frame, makes a square of side-length 80 cm. The frame is 4 cm wide.

What percentage of the area of the square is covered by the frame?

A 15% B 18% C 19% D 20% E 24%



Solution **C**

The horizontal edges of the frame are rectangles with dimensions $4\,\text{cm} \times 80\,\text{cm}$. Hence they each have area $(4 \times 80)\,\text{cm}^2$.

The vertical edges of the frame are rectangles with dimensions $4\,\text{cm} \times (80 - 2 \times 4)\,\text{cm}$, that is, $4\,\text{cm} \times 72\,\text{cm}$. Hence they each have area $(4 \times 72)\,\text{cm}^2$.

Therefore the total area of the frame is $\big(2 \times (4 \times 80) + 2 \times (4 \times 72)\big)\,\text{cm}^2 = \big(2 \times 4 \times (80 + 72)\big)\,\text{cm}^2$.

The area of the square is $(80 \times 80)\,\text{cm}^2$.

Therefore the percentage of the area of the square covered by the frame is

$$\left(\frac{2 \times 4 \times (80 + 72)}{80 \times 80}\right) \times 100\% = \left(\frac{10 + 9}{10 \times 10}\right) \times 100\% = 19\%.$$

Note that in the above calculation we have resisted the temptation to evaluate $2 \times 4 \times (80 + 72)$ and 80×80 separately. This enables us to save a lot of work at the final stage by dividing the numerator and denominator by common factors. This process is often called *cancelling* common factors.

Thus here $2 \times 4 = 8 = 1 \times 8$ and $80 = 10 \times 8$. So, by cancelling the common factor 8, we have $\dfrac{2 \times 4}{80} = \dfrac{1}{10}$.

Also, $80 + 72 = 10 \times 8 + 9 \times 8$. So, again by cancelling the common factor 8, we have $\dfrac{(80 + 72)}{80} = \dfrac{10 + 9}{10}$.

It follows that

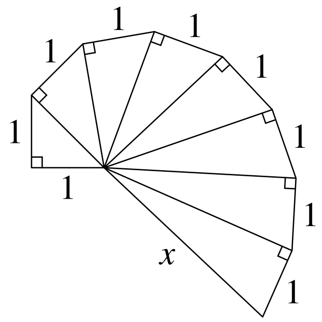
$$\left(\frac{2 \times 4 \times (80 + 72)}{80 \times 80}\right) = \left(\frac{2 \times 4}{80}\right) \times \left(\frac{80 + 72}{80}\right) = \left(\frac{1}{10}\right) \times \left(\frac{10 + 9}{10}\right) = \left(\frac{10 + 9}{10 \times 10}\right) = \frac{19}{100}.$$

For investigation

11.1 What percentage of the area of the square would be covered by the frame if the frame were 8 cm wide?

11.2 How wide would the frame need to be in order to cover exactly 10% of the square?

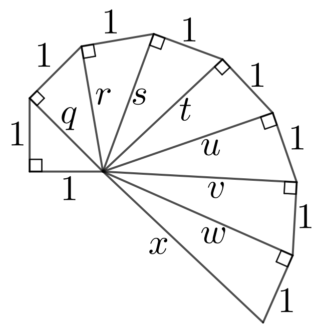
12. What is the length of the line segment marked x?



A $\sqrt{2}$ B 2 C $2\sqrt{2}$ D 3 E 4

Solution **D**

We let q, r, s, t, u, v, w and x be the lengths of the line segments, as shown in the diagram on the right.



q is the length of the hypotenuse of a right-angled triangle whose shorter sides each have length 1.

Therefore, by Pythagoras' theorem, $q^2 = 1^2 + 1^2 = 2$.

Similarly, r is the length of the hypotenuse of a right-angled triangle whose shorter sides have lengths 1 and q. Therefore, by Pythagoras' theorem, $r^2 = 1^2 + q^2 = 1 + 2 = 3$.

In a similar way, we have

$$\begin{aligned} s^2 &= 1 + r^2 = 4 \\ t^2 &= 1 + s^2 = 5 \\ u^2 &= 1 + t^2 = 6 \\ v^2 &= 1 + u^2 = 7 \\ w^2 &= 1 + v^2 = 8 \end{aligned}$$

and, finally,

$$x^2 = 1 + w^2 = 9.$$

We therefore conclude that $x = \sqrt{9} = 3$.

For investigation

12.1 What is the area of the shape in this question?

> The diagram in this question can be extended indefinitely by adding right-angled triangles of which one side adjacent to the right angle is the hypotenuse of the previous triangle and the other has length 1. When this is done, the line made up of the sides of length 1 is called the *Spiral of Theodorus.* It is named after Theodorus of Cyrene (465 BCE - 398 BCE). Cyrene is in modern day Libya.

12.2 How many triangles can you add in this way before the last triangle you construct overlaps the first triangle?

12.3 Search the internet to see what information you can find about the Spiral of Theodorus.

13. When I increase a certain number by 20%, I get twice as much as when I decrease 20 less than the number by 20%.

What is that number?

A 40 B 60 C 80 D 100 E 120

Solution **C**

The most straightforward method is just to try out the options in turn until you find an option that works. This is our first method. Our second method uses algebra. This is the type of method you would have needed to use if no options had been given, as in Problem 13.1.

Method 1

x	$y = x + 20\%$ of x	$x - 20$	$z = (x - 20) - 20\%$ of $x - 20$	$y = 2z$?
40	48	20	16	no
60	72	40	32	no
80	96	60	48	yes

We see that 80 is the correct option.

Method 2

Let the number in question be x.

When I increase x by 20%, I obtain the number $\frac{120}{100}x$, that is $\frac{6}{5}x$.

When I decrease 20 less than x by 20%, I obtain the number $\frac{80}{100}(x - 20)$, that is, $\frac{4}{5}(x - 20)$.

Because the first number is twice the second, $\frac{6}{5}x = 2\left(\frac{4}{5}(x - 20)\right)$.

By expanding the right-hand side of this equation, we obtain $\frac{6}{5}x = \frac{8}{5}x - 32$.

This last equation may be rearranged to give $\frac{2}{5}x = 32$.

Therefore $x = \frac{5}{2} \times 32 = 5 \times 16 = 80$.

For investigation

13.1 When I increase another number by 25%, I get three times as much as when I decrease 16 less than the number by 25%.

What is this other number?

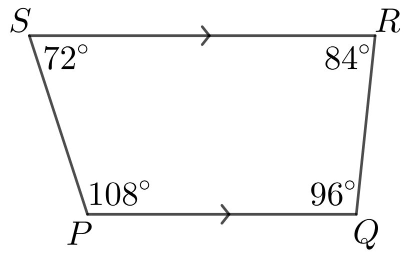
14. Going clockwise around a quadrilateral, its interior angles are in the ratio 6 : 7 : 8 : 9.

Which of the following is a true statement about the quadrilateral?

A It has a right angle B It is a trapezium C It is a parallelogram
D It is a kite E It is cyclic

SOLUTION **B**

The sum of the angles in a quadrilateral is 360°.



Because 6 + 7 + 8 + 9 = 30, the angles of the quadrilateral which are in the ratio 6 : 7 : 8 : 9 are $\frac{6}{30} \times 360°$, $\frac{7}{30} \times 360°$, $\frac{8}{30} \times 360°$ and $\frac{9}{30} \times 360°$.

That is, the angles are 72°, 84°, 96° and 108°, as shown in the diagram, where we have labelled the vertices P, Q, R and S.

We see that $\angle PSR + \angle QPS = 72° + 108° = 180°$. It follows that SR is parallel to PQ. On the other hand $\angle PSR + \angle SRQ = 72° + 84° \neq 180°$. Therefore SP is not parallel to RQ.

We have seen that $PQRS$ is a quadrilateral with just one pair of parallel sides. It follows that $PQRS$ is a trapezium.

In the context of the IMC, we can stop here, having found that $PQRS$ is a trapezium. For a complete answer it is necessary to show that none of the other options is correct. We have already seen that option A is not correct. In Problem 14.1, you are asked to explain why options C, D and E are not correct.

FOR INVESTIGATION

14.1 Explain why $PQRS$ is neither a parallelogram, nor a kite, nor a cyclic quadrilateral.

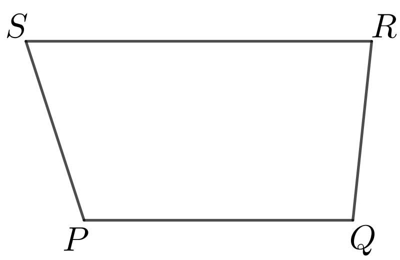
14.2 Suppose that, going clockwise around the quadrilateral $PQRS$, the angles are in the ratio $p : q : r : s$.

For which of the following shapes is there an equation, or a set of equations, relating the values of p, q, r and s which implies that $PQRS$ has that shape?

(a) trapezium, (b) parallelogram, (c) cyclic quadrilateral, (d) kite,

(e) rectangle, (f) square, (g) rhombus.

14.3 The solution of this question uses the following facts about a quadrilateral $PQRS$.



(a) If $\angle PSR + \angle QPS = 180°$, then SR is parallel to PQ.

(b) If $\angle PSR + \angle SRQ \neq 180°$, then SP is not parallel to RQ.

Prove that both (a) and (b) are true.

15. Carrie the cat and Barrie the bat together weigh 4000 g more than Rollie the rat.

Barrie and Rollie together weigh 2000 g less than Carrie.

Carrie and Rollie together weigh 3000 g more than Barrie.

What is the weight, in grams, of Rollie the rat?

A 250 B 500 C 750 D 1000 E 1250

SOLUTION **B**

We let c, b and r be the weights, in grams, of Carrie, Barrie and Rollie respectively.

The information given in the question yields the following equations.

$$c + b = r + 4000 \qquad (1)$$
$$b + r = c - 2000 \qquad (2)$$
$$c + r = b + 3000 \qquad (3)$$

Adding equations (2) and (3) gives

$$c + b + 2r = c + b + 1000$$

from which we deduce, by subtracting $c + b$ from both sides,that

$$2r = 1000.$$

Therefore

$$r = 500.$$

Hence Rollie's weight in grams, is 500.

FOR INVESTIGATION

15.1 What are the weights of Carrie and Barrie?

15.2 You are given that

$$x + y + z = w + 100,$$
$$w + y + z = x + 200,$$
$$w + x + z = y + 300,$$
$$\text{and} \quad w + x + y = z + 400.$$

Find the values of w, x, y, and z.

16. Factorial n, written $n!$, is defined by: $n! = 1 \times 2 \times 3 \times \cdots \times n$.

What is the remainder when $1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9! + 10!$ is divided by 5?

A 0 B 1 C 2 D 3 E 4

SOLUTION **D**

We can work out the remainder when an integer is divided by 5 from its units (ones) digit.

We have

$$\begin{aligned} 1! &= 1, \\ 2! &= 1 \times 2 = 2, \\ 3! &= 1 \times 2 \times 3 = 6, \\ 4! &= 1 \times 2 \times 3 \times 4 = 24, \\ 5! &= 1 \times 2 \times 3 \times 4 \times 5 = 120. \end{aligned}$$

We see that $5!$ has 0 as its units digit. Similarly, for $n > 5$, $n! = 1 \times \mathbf{2} \times 3 \times 4 \times \mathbf{5} \times \cdots \times n$ and is therefore a multiple of 10. Hence $n!$ also has units digit 0 for $n > 5$.

It follows that the units digit of $1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9! + 10!$ is the same as the units digit of $1! + 2! + 3! + 4!$. Now $1! + 2! + 3! + 4! = 1 + 2 + 6 + 24 = 33$. Hence its units digit is 3.

Therefore the units digit of $1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9! + 10!$ is also 3. Hence the remainder when this number is divided by 5 is 3

FOR INVESTIGATION

16.1 What is the remainder when $1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9! + 10!$ is divided by 7?

17. What is $4^{(3^2)}$ divided by $(4^3)^2$?

A 1 B 6 C 16 D 32 E 64

SOLUTION **E**

We have $4^{(3^2)} = 4^9$ and $(4^3)^2 = 4^{3 \times 2} = 4^6$.

Therefore

$$4^{(3^2)} \div (4^3)^2 = \frac{4^9}{4^6} = 4^{9-6} = 4^3 = 64.$$

FOR INVESTIGATION

17.1 Find the values, in the form 5^k, where k is an integer, of

(a) $5^{(4^3)} \div (5^4)^3$, (b) $5^{(4^{(3^2)})} \div ((5^4)^3)^2$.

17.2 Explain why if m and n are positive integers, with $m > n$, then for all non-zero numbers x, we have $x^m \div x^n = x^{m-n}$.

18. The point $P\,(-1, 4)$ is reflected in the y-axis to become Q. The point Q is reflected in the line $y = x$ to become R. The point R is reflected in the x-axis to become S.

What is the area of quadrilateral $PQRS$?

A 4 B $4\sqrt{2}+2$ C 6 D $4+2\sqrt{2}$ E 8

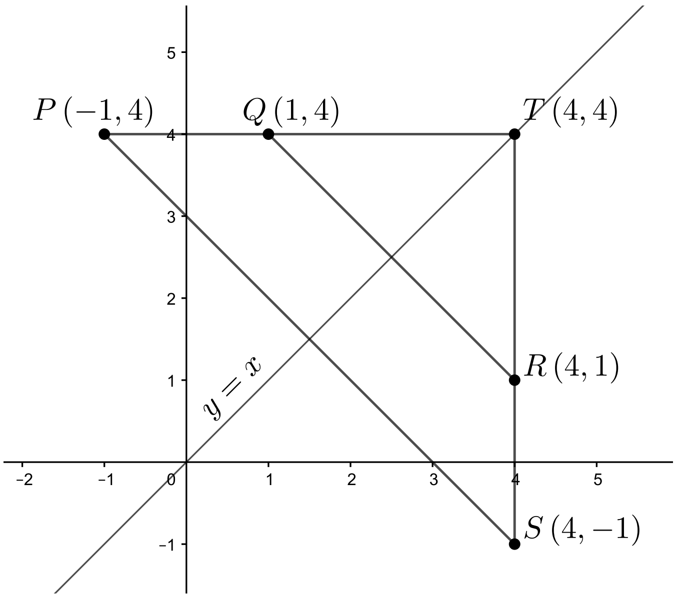
SOLUTION **E**

The reflection in the y-axis transforms the point $P\,(-1, 4)$ into the point $Q\,(1, 4)$.

The reflection in the line $y = x$ transforms the point $Q\,(1, 4)$ into the point $R\,(4, 1)$.

The reflection in the x-axis, transforms the point $R\,(4, 1)$ into the point $S\,(4, -1)$.

We now see that the area of the quadrilateral $PQRS$ is the area of the triangle PTS minus the area of the triangle QTR, where $T\,(4, 4)$ is the point at which the line through P and Q meets the line through R and S.



Hence the area of $PQRS$ is given by

$$\begin{aligned}\tfrac{1}{2}(PT \times TS) - \tfrac{1}{2}(QT \times TR) &= \tfrac{1}{2}(5 \times 5) - \tfrac{1}{2}(3 \times 3)\\ &= \tfrac{1}{2}(25 - 9)\\ &= \tfrac{1}{2}(16)\\ &= 8.\end{aligned}$$

FOR INVESTIGATION

18.1 Note that $PQRS$ is a trapezium. Use the formula for the area of a trapezium to find the area of $PQRS$.

18.2 The point $P\,(2, -3)$ is reflected in the line $y = x$ to become Q. The point Q is reflected in the y-axis to become R. The point R is reflected in the x-axis to become S.

What is the area of the quadrilateral $PQRS$?

18.3 Suppose a and b are positive numbers with $a < b$.

The point $P\,(a, b)$ is reflected in the y-axis to become Q. The point Q is reflected in the line $y = x$ to become R. The point R is reflected in the x-axis to become S.

Find the area of the quadrilateral $PQRS$ in terms of a and b.

19. In the grid shown the three non-zero numbers in each row, each column and each diagonal *multiply* to give the same *product*.

What is the value of x?

A 36 B 18 C 12 D 9 E 4

	6	
2	x	3

SOLUTION **A**

From the bottom row we see that the common value of the products is $2 \times x \times 3 = 6x$.

Let the numbers in the cells in the top row be r, s and t from left to right.

Then the product of the numbers in the diagonal from top-left to bottom-right is $r \times 6 \times 3 = 18r$. It follows that $18r = 6x$. Hence $r = x/3$.

$x/3$	1	$x/2$
	6	
2	x	3

The product of the numbers in the middle column is $s \times 6 \times x = 6sx$. It follows that $6sx = 6x$. Hence, as $x \neq 0$, $s = 1$.

The product of the numbers in the diagonal from top-right to bottom-left is $t \times 6 \times 2 = 12t$. It follows that $12t = 6x$. Hence $t = x/2$.

The product of the numbers in the top row is $x/3 \times 1 \times x/2 = x^2/6$. It follows that $x^2/6 = 6x$. Hence, as $x \neq 0$, we can deduce that $x/6 = 6$. Therefore $x = 36$.

FOR INVESTIGATION

19.1 Find all the other numbers in the grid.

20. A shop sign says, "T-shirts. Three for the price of two. Equivalent to a saving of £5.50 on each T-shirt."

Using this special offer, what is the cost of three T-shirts?

A £16.50 B £22 C £31 D £33 E £49.50

SOLUTION **D**

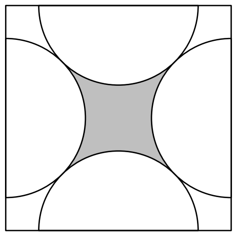
Let the full price of a T-shirt be £t. Then the difference between the full price of three T-shirts and the price when you buy three for the price of two is £$3t$ – £$2t$ = £t. Since this is equivalent to a saving of £5.50 on each of the three T-shirts, £t = $3 \times$ £5.50 = £16.50.

Using the special offer, the amount you pay for three T-shirts is the full price of two T-shirts, that is, $2 \times$ £16.50 = £33.

FOR INVESTIGATION

20.1 The shop next door, not to be outdone, advertises "Save £10 per T-shirt, when you buy four for the price of three!!!". When you buy four T-shirts in this shop, how much are you paying per T-shirt? How does this compare with the first shop?

21. The diagram shows a square of side 4 cm with four identical semi-circles drawn with their centres at the mid-points of the sides. The four semi-circles each touch two other semi-circles, as shown.

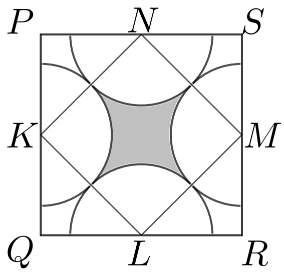


What is the shaded area, in cm^2?

A $8 - \pi$ B π C $\pi - 2$ D $\pi - \sqrt{2}$ E $8 - 2\pi$

SOLUTION **E**

We let the vertices of the square be P, Q, R and S, and the midpoints of the edges be K, L, M and N, as shown.



$KLMN$ is a square. You are asked to prove this in Problem 21.2.

The square $PQRS$ has sides of length 4 cm. Hence $KQ = QL = 2$ cm.

Therefore, by Pythagoras' Theorem, the length of KL is $\sqrt{2^2 + 2^2}$ cm $= 2\sqrt{2}$ cm.

The part of each semicircle that is inside the square $KLMN$ is a quarter circle.

The shaded area is the area of the square $KLMN$ minus the areas of these four quarter circles.

The area of the square $KLMN$ is KL^2, that is, $(2\sqrt{2})^2\,\text{cm}^2 = 8\,\text{cm}^2$.

The length, $2\sqrt{2}$ cm, of KL is the the sum of the lengths of two radii of the semicircles. It follows that the radius of the semicircles is $\sqrt{2}$ cm.

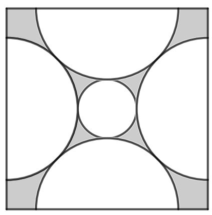
The four quarter circles make up a circle of radius $\sqrt{2}$ cm and hence area $\pi\sqrt{2}^2\,\text{cm}^2 = 2\pi\,\text{cm}^2$.

Therefore the shaded area is

$$8\,\text{cm}^2 - 2\pi\,\text{cm}^2 = (8 - 2\pi)\,\text{cm}^2.$$

FOR INVESTIGATION

21.1 The diagram on the right has been obtained by adding to the diagram of this question a circle which touches all four semicircles.



The region inside the square but not inside the semicircles, nor inside the circle, has been shaded.

What is the shaded area, in cm^2?

21.2 Prove that $KLMN$ is a square.

22. When a cube is cut into two pieces with a single plane cut, two polyhedra are obtained.

Which of these polyhedra *cannot* be obtained in this way?

A A polyhedron with 4 faces
B A polyhedron with 5 faces
C A polyhedron with 6 faces
D A polyhedron with 7 faces
E A polyhedron with 8 faces

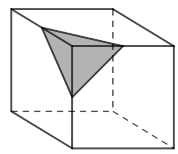
Solution **E**

When a polyhedron is formed by a plane cut through a cube, it will have one face created by the cut. Its other faces will be formed from all or part, of some or all, of the six faces of the cube.

It follows that the polyhedron can have at most one more face than the cube, that is, at most seven faces. Therefore it is not possible to obtain a polyhedron with eight faces.

In the context of the IMC the argument above is enough to show that the correct option is E. However, for a complete solution it is necessary to rule out the other options by giving suitable examples. This we now do.

The diagram on the right shows a cube with a plane cut through points on three adjacent edges of the cube.

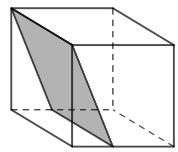


It can be seen that this produces a tetrahedron which has four faces, and a second polyhedron.

The second polyhedron has three square faces which are faces of the original cube, three pentagonal faces formed by removing a triangular corner from the other three faces of the cube, and the triangular face created by the cut. It is therefore a polyhedron with seven faces.

This shows that neither option A nor D is correct.

The diagram on the right shows a plane cut through two adjacent vertices and two points on the bottom edges of a cube. One of the polyhedra created by this cut is a triangular prism. This has five faces.



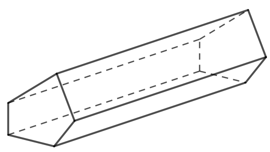
The other polyhedron is a prism whose end faces are trapeziums. This has six faces.

This shows that neither option B nor C is correct.

For investigation

22.1 (a) Consider a pentagonal prism, that is, a prism where the polygons that form the end faces are pentagons.

How many faces does such a prism have?



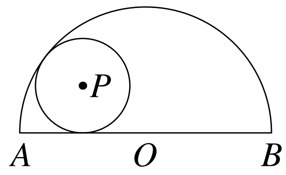
(b) Suppose that the polygons that form the end faces of a prism have n edges.

How many faces does such a prism have?

22.2 What are the possible number of faces of a polyhedron formed by making a plane cut through a tetrahedron?

23. A circle is inscribed in a semicircle with centre O and diameter AB. The centre of the circle is the point P, where $PA = PO$.

What is the ratio of the radius of the circle to the radius of the semicircle?

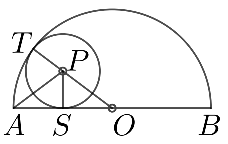


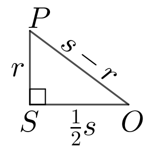
A 4 : 9 B 3 : 8 C 3 : 7 D 2 : 5 E 1 : 2

Solution **B**

We let s be the radius of the semicircle and r be the radius of the circle.

We let S and T be the points where the semicircle touches the circle, as shown.





Because the diameter AB is a tangent to the circle at S, $\angle PSA = \angle PSO = 90°$. Therefore PSA and PSO are right-angled triangles in which $PA = PO$ and the side PS is common. Hence the triangles PSA and PSO are congruent (RHS). It follows that $OS = AS$

Since OA is a radius of the semicircle, $OA = s$. Therefore $OS = \frac{1}{2}OA = \frac{1}{2}s$.

Because the circle touches the semicircle at T, they have a common tangent at T. The radii PT and OT are perpendicular to this common tangent. Therefore the points O, P and T are collinear. Hence $OP = OT - PT = s - r$.

PS is a radius of the circle. Therefore $PS = r$.

We can now apply Pythagoras' Theorem to the right-angled triangle PSO to give

$$r^2 + (\tfrac{1}{2}s)^2 = (s - r)^2.$$

Expanding, we obtain

$$r^2 + \tfrac{1}{4}s^2 = s^2 - 2rs + r^2.$$

Hence

$$2rs = \tfrac{3}{4}s^2.$$

Since $s \neq 0$, we can deduce that $r = \frac{3}{8}s$. Therefore $r : s = 3 : 8$.

By a *Pythagorean Triangle* we mean a right-angled triangle in which the ratios of the side lengths are all rational numbers.

Note that in this question, because $r : s = 3 : 8$, in the triangle PSO we have

$$PS : SO : OP = r : \tfrac{1}{2}s : s - r = 3 : 4 : 5.$$

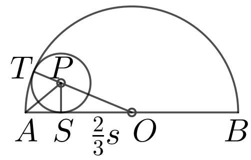
Thus PSO is the most basic of all Pythagorean triangles.

In problem 23.1 you are asked to explore which other Pythagorean triangles arise when a circle touches a semi-circle.

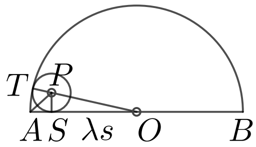
FOR INVESTIGATION

23.1 (a) Suppose that $SO = \frac{2}{3}s$.

Find the ratios of the side lengths in the right-angled triangle PSO in this case.



(b) Find the value of λ in the case where $SO = \lambda s$ and $PS : SO : OP = 7 : 24 : 25$.

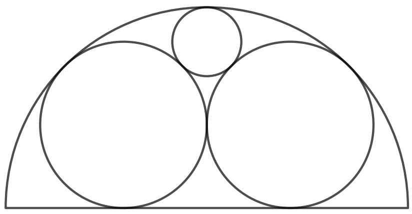


(c) Suppose that $SO = \lambda s$. Find the ratios of the side lengths in the right-angled triangle PSO in terms of λ.

Deduce that when λ is a rational number, the triangle PSO is Pythagorean.

(d) Give an example of a right-angled triangle in which the ratios of the side lengths are not rational numbers.

23.2



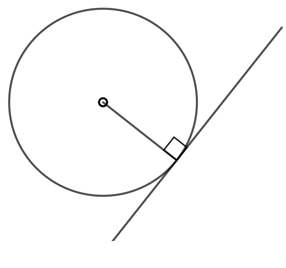
The diagram shows a semicircle, a small circle and two larger circles which have the same radius, which touch each other, as shown.

(a) Find the ratio of the radius of the larger circles to the radius of the semicircle in the form $1 : x$.

(b) Find the ratio of the radius of the smaller circle to the radius of the larger circles, in the form $1 : y$.

23.3 In the solution to Problem 23 we have used the following:

Theorem
A radius of a circle is perpendicular to the tangent to the circle at the point where the radius meets the circle.

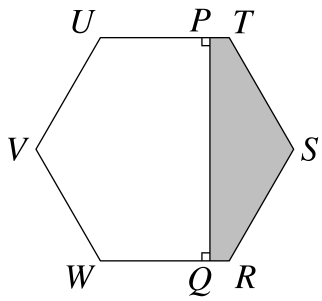


Find a proof of this theorem, either by proving it for yourself, or by looking in a book or on the internet, or by asking your teacher.

24. The diagram shows a regular hexagon $RSTUVW$.

The area of the shaded pentagon $RSTPQ$ is one quarter of the area of the hexagon. Jay and Kay walk around the hexagon from P to Q, Jay going clockwise and Kay anticlockwise.

What is the ratio of the distance Jay walks to the distance Kay walks?



A 1 : 2 B 2 : 3 C 3 : 5 D 4 : 7 E 5 : 8

Solution **C**

Note that this question gives us information about *areas* and asks us to deduce information about *lengths*. Usually it is the other way round.

In their walks both Jay and Kay walk along two edges of the regular hexagon. The difference in the lengths that they walk arises because Jay walks along PT and RQ, whereas Kay walks along PU and WQ. Therefore, in order to work out the ratios of the distances that they walk, we need to find the lengths of PT and PU as fractions of the length of the sides of the hexagon.

We now note that the rectangles $QRTP$ and $WRTU$ have the same height given by the length of TR, but different widths. The width of $QRTP$ is given by the length of PT, and the width of $WRTU$ is given by the length of TU.

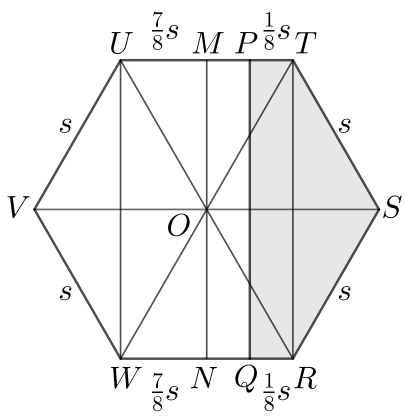
Therefore, we can find the ratio $PT : TU$ from the ratio of the areas of the two rectangles. We can find the ratio of these areas from the information we are given that the area of $RSTPQ$ is one quarter of the area of the hexagon.

Note in the solution which follows the argument is given in the reverse of the order in which we thought about the problem.

This is the the logical order because it begins with what we are told in the question about the ratio of the areas and deduces the solution.

It is usual and, on the whole, helpful to the reader to present solutions in this logical order. However, it has the disadvantage that it hides the way in which the solution was found.

Let O be the centre of the hexagon, and let M and N be the midpoints of UT and WR, respectively. We also let s be the sidelength of the hexagon.



The three diagonals through O together with the lines UW, MN and TR divide the hexagon into twelve congruent triangles. Let t be the area of one of these triangles.

Then the area of the hexagon is $12t$.

The area of the shaded pentagon $RSTPQ$ is one quarter of this. So its area is $3t$.

The pentagon $RSTPQ$ is made up of the triangle RST and the rectangle $QRTP$. We see from the diagram that the area of RST is $2t$. Hence the area of the rectangle $QRTP$ is t.

It may also be seen from the diagram that the area of the rectangle $WRTU$ is $8t$. We therefore have

$$\frac{PT}{TU} = \frac{PT \times TR}{TU \times TR} = \frac{\text{area } QRTP}{\text{area } WRTU} = \frac{t}{8t} = \frac{1}{8}.$$

Thus the length of PT is $\frac{1}{8}s$ and the length of UP is $\frac{7}{8}s$. Similarly, the length of RQ is $\frac{1}{8}s$, and that of WQ is $\frac{7}{8}s$.

Hence the distance that Jay walks clockwise along PT, TS, SR and RQ is $\frac{1}{8}s + s + s + \frac{1}{8}s = \frac{9}{4}s$.

Similarly, the distance Kay walks anticlockwise along PU, UV, VW and WQ is $\frac{7}{8}s+s+s+\frac{7}{8}s = \frac{15}{4}s$.

Therefore the ratio of the distance Jay walks to the distance Kay walks is

$$\tfrac{9}{4} : \tfrac{15}{4} = 9 : 15 = 3 : 5.$$

FOR INVESTIGATION

24.1 The solution above makes frequent use of the fact that the hexagon has been divided into 12 triangles which are all congruent, and hence have the same area.

Prove that these triangles are all congruent.

24.2 Explain how it follows from the diagram that the area of the triangle RST is $2t$, and the area of the rectangle $WRTU$ is $8t$.

24.3 Suppose that the area of the pentagon $PQRTS$ is five-twelfths of the area of the hexagon.

What is the ratio of the distance Jay walks to the distance Kay walks in this case?

24.4 Suppose that the ratio of the distance Jay walks to the distance Kay walks is 2 : 3.

What is the ratio

$$\text{area of the pentagon } RSTPQ : \text{area of the hexagon } RSTUVW$$

in this case?

24.5 Suppose that

$$\text{area of the pentagon } RSTPQ : \text{area of the hexagon } RSTUVW = \lambda : 1,$$

where $\frac{1}{6} \leq \lambda \leq \frac{5}{6}$.

Find the ratio

$$\text{distance Jay walks} : \text{distance Kay walks}$$

in terms of λ.

24.6 This problem uses a different method to show that $PT = \frac{1}{8}s$.

(a) Find the areas of the hexagon $RSTUVW$ and the triangle TRS in terms of s.

(b) Deduce the areas of the pentagon $RSTPQ$ and the rectangle $QRTP$ in terms of s.

(c) Find the length of TR in terms of s.

(d) Deduce that $PT = \frac{1}{8}s$.

25. A gold coin is worth x% more than a silver coin. The silver coin is worth y% less than the gold coin. Both x and y are positive integers.

How many possible values for x are there?

A 0 B 3 C 6 D 9 E 12

Solution **E**

Let the values of a gold coin and a silver coin be g ducats and s ducats, respectively.

Since a gold coin is worth x% more than a silver coin, $g = \frac{100 + x}{100} \times s$. Hence $\frac{g}{s} = \frac{100 + x}{100}$.
Since a silver coin is worth y% less than a gold coin, $s = \frac{100 - y}{100} \times g$. Hence $\frac{g}{s} = \frac{100}{100 - y}$.

Therefore $\frac{100 + x}{100} = \frac{100}{100 - y}$. Hence $(100 + x)(100 - y) = 100 \times 100 = 10\,000$.

Because x is a positive integer, it follows that $100 + x$ is a factor of $10\,000$ with $100 + x > 0$.

Therefore we need to count the factors of 10 000 which are greater than 100.

Method 1

In this method we just list the factors that are greater than 100, and then count them.

The factors of 10 000 which are greater than 100 are 125, 200, 250, 400, 500, 625, 1000, 1250, 2000, 2500, 5000 and 10 000. So there are 12 possible values for $100 + x$, Hence there are 12 possible values of x, namely, 25, 100, 150, 300, 400, 525, 900, 1150, 1900, 2400, 4900 and 9900.

Method 2

In this method we work out the number of factors, without listing them, by considering the factorization of 10 000 into primes.

The factorization of 10 000 into primes is $2^4 \times 5^4$. Therefore 10 000 has the 25 factors $2^m \times 5^n$ for $0 \leq 4 \leq m$ and $0 \leq n \leq 4$.

One of these factors is 100, with $10\,000 = 100 \times 100$. The other 24 factors occur in 12 pairs. Each pair consists of one factor greater than 100 and one factor less than 100.

Hence 10 000 has 12 factors greater than 100. Therefore there are 12 possible values of $100 + x$ and hence 12 possible values for x.

For investigation

25.1 How many factors does 1 000 000 have? How many of these factors are greater than 1000?

25.2 Find formulas, in terms of n, for the number of factors of 10^{2n} and the number of these factors that are greater than 10^n.